Supplementary material for the paper: Asymptotic velocities in quantum and Bohmian mechanics, by Bruno Galvan

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In this document an example is given of a relativistic trajectory space which is not asymptotically regular but nevertheless $S_t \to S_\infty$ for $t \to \infty$, and it is shown that gS_t does not converge to gS_∞, where $gS_t := gP_t \circ \eta_t|^{-1}_{g\mathbb{B}_A}$ and $gS_\infty(V) := S_\infty(g^{-1}V)$.

Let us define first the trajectory

$$\mathbf{k}_\varphi(t) := R(\hat{\mathbf{n}}, \ln \omega t)\mathbf{v}_\varphi t := \mathbf{v}_\varphi(t)t,$$

where $\hat{\mathbf{n}} = (0,0,1)$, $\mathbf{v}_\varphi = v(\cos\varphi, \sin\varphi, 0)$, with $0 < v < 1/\sqrt{2}$. By continuity we can define $\mathbf{k}_\varphi(0) := 0$. For simplicity, in this document all the trajectories will be considered as defined on $\mathbb{R}^+$ instead of $\mathbb{R}$. Note that

$$\mathbf{v}_\varphi(t) = v(\cos(\varphi + \ln \omega t), \sin(\varphi + \ln \omega t), 0).$$

The logarithm has been introduced in order to guarantee that $\mathbf{k}_\varphi$ is a world line; in fact the instantaneous velocity of $\mathbf{k}_\varphi$ is

$$\dot{\mathbf{k}}_\varphi(t) = \hat{\mathbf{n}} \wedge \mathbf{v}_\varphi(t) + \mathbf{v}_\varphi(t),$$

so that

$$\|\dot{\mathbf{k}}_\varphi(t)\|^2 \leq 2\|\mathbf{v}_\varphi(t)\|^2 < 1.$$

The trajectory $\mathbf{k}_\varphi$ does not admit asymptotic velocity because $\eta_t(\mathbf{k}_\varphi) = \mathbf{v}_\varphi(t)$.

Let us consider now the trajectory space composed of the set of trajectories

$$\mathbb{B} := \{(\mathbf{k}_\varphi, \mathbf{k}_{\varphi+\pi}) : \varphi \in [0, 2\pi)\}, \tag{1}$$

endowed with the measure induced by the normalized Lebesgue measure on $[0, 2\pi)$. The support of the measure S_t is the set

$$\begin{aligned}
&\eta_t(\mathbb{B}) = \{\\
&(v\cos(\varphi + \ln\omega t), v\sin(\varphi + \ln\omega t), 0, -v\cos(\varphi + \ln\omega t), -v\sin(\varphi + \ln\omega t), 0) :\\
&: \varphi \in [0, 2\pi)\}
\end{aligned}$$

or equivalently

$$\eta_t(\mathbb{B}) = \{(v\cos\varphi, v\sin\varphi, 0, -v\cos\varphi, -v\sin\varphi, 0) : \varphi \in [0, 2\pi)\}. \tag{2}$$

One can easily see that S_t is the uniform (Lebesgue) measure concentrated on $\eta_t(\mathbb{B})$, as in the example at the end of the section 2 of the paper. The measure S_t is therefore independent of time and $S_t \to S_\infty = S_t$ for $t \to \infty$. Of course the support of S_∞ is the set (2); if g is a boost of velocity $u < 1$ along the x-axis, the support of gS_∞ is therefore the set

$$g\eta_+(\mathbb{B}) = \left\{ \left(\frac{v\cos\varphi - u}{1 - uv\cos\varphi}, \frac{v\sin\varphi}{\gamma(1 - uv\cos\varphi)}, 0, \frac{-v\cos\varphi - u}{1 + uv\cos\varphi}, \frac{-v\sin\varphi}{\gamma(1 + uv\cos\varphi)}, 0 \right) : \varphi \in [0, 2\pi) \right\}. \tag{3}$$

We will study now the support of the measure $gS_t = gP \circ \eta_+|_{g\mathbb{B}}^{-1}$, and we will show that, if the measure converges for $t \to \infty$, the support of its limit cannot be (3), and therefore $\lim_{t\to\infty} gS_t \neq gS_\infty$.

First of all let us study how does $\mathbf{k}_\varphi$ transform. The graph of $\mathbf{k}_\varphi$ is

$$\overline{\mathbf{k}}_\varphi = \{(1, v\cos(\varphi + \ln\omega t), v\sin(\varphi + \ln\omega t), 0)t : t > 0\},$$

By defining $\theta := \ln\omega t$ we can write

$$\overline{\mathbf{k}}_\varphi = \{(1, v\cos(\varphi + \theta), v\sin(\varphi + \theta), 0)e^\theta/\omega : \theta \in \mathbb{R}\}.$$

The transformed graph is:

$$g\overline{\mathbf{k}}_\varphi = \{(\gamma[1 - uv\cos(\varphi + \theta)], \gamma[v\cos(\varphi + \theta) - u], v\sin(\varphi + \theta), 0)e^\theta/\omega : \theta \in \mathbb{R}\}.$$

Let

$$\tilde{s}(\varphi, \theta) := \frac{e^\theta\gamma}{\omega}[1 - uv\cos(\varphi + \theta)].$$

Note that $\tilde{s}(\varphi, \theta) \to 0$ for $\theta \to -\infty$ and $\tilde{s}(\varphi, \theta) \to \infty$ for $\theta \to \infty$. Note moreover that

$$\tilde{s}(\varphi, \theta + 2n\pi) = e^{2n\pi}\tilde{s}(\varphi, \theta) \text{ for } n = 1, 2, \ldots. \tag{4}$$

The θ-derivative of $\tilde{s}(\varphi, \theta)$ is greater than 0:

$$\partial_\theta\tilde{s}(\varphi, \theta) = \frac{e^\theta\gamma}{\omega}[1 - uv\cos(\varphi+\theta)] + \frac{e^\theta\gamma}{\omega}uv\sin(\varphi+\theta) = \frac{e^\theta\gamma}{\omega}\{1 + uv[\sin(\varphi+\theta) - \cos(\varphi+\theta)]\} > 0$$

because $uv < 1/\sqrt{2}$ and $\sin\theta - \cos\theta \geq -2/\sqrt{2}$. The function $\tilde{s}(\varphi, \cdot)$ is therefore invertible, and let $\tilde{\theta}(\varphi, \cdot) := \tilde{s}^{-1}(\varphi, \cdot)$; note that $\tilde{\theta}(\varphi, \cdot)$ is increasing in the interval $(0, \infty)$, with $\lim_{s\to 0}\tilde{\theta}(\varphi, s) = -\infty$ and $\lim_{s\to\infty}\tilde{\theta}(\varphi, s) = \infty$. Moreover, from the equality (4) one easily deduces the equality

$$\tilde{\theta}(\varphi, s) + 2n\pi = \tilde{\theta}(\varphi, se^{2n\pi}) \text{ for } n = 1, 2, \ldots. \tag{5}$$

that will be utilized later. We can therefore write

$$g\overline{\mathbf{k}}_\varphi = \{(\gamma[1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))], \gamma[v\cos(\varphi+\tilde{\theta}(\varphi,s))-u], v\sin(\varphi+\tilde{\theta}(\varphi,s)), 0)e^{\tilde{\theta}(\varphi,s)}/\omega : s>0\}.$$

Since, by definition,

$$s = \frac{e^{\tilde{\theta}(\varphi,s)}\gamma}{\omega}[1 - uv\cos(\varphi+\tilde{\theta}(\varphi,s))],$$

we can also write

$$g\overline{\mathbf{k}}_\varphi = \left\{\left(1, \frac{v\cos(\varphi+\tilde{\theta}(\varphi,s))-u}{1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))}, \frac{v\sin(\varphi+\tilde{\theta}(\varphi,s))}{\gamma[1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))]}, 0\right) s : s>0\right\},$$

and therefore the transformed trajectory $g\mathbf{k}_\varphi$ is

$$g\mathbf{k}_\varphi(s) = \left(\frac{v\cos(\varphi+\tilde{\theta}(\varphi,s))-u}{1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))}, \frac{v\sin(\varphi+\tilde{\theta}(\varphi,s))}{\gamma[1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))]}, 0\right) s.$$

Let us study now the set $g\mathbb{B}$, where $\mathbb{B}$ is defined by definition (1). We have:

$$\eta_s(g\mathbb{B}) = \Big\{\Big(\frac{v\cos(\varphi+\tilde{\theta}(\varphi,s))-u}{1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))}, \frac{v\sin(\varphi+\tilde{\theta}(\varphi,s))}{\gamma[1-uv\cos(\varphi+\tilde{\theta}(\varphi,s))]}, 0,$$
$$\frac{-v\cos(\varphi+\tilde{\theta}(\varphi+\pi,s))-u}{1+uv\cos(\varphi+\tilde{\theta}(\varphi+\pi,s))}, \frac{-v\sin(\varphi+\tilde{\theta}(\varphi+\pi,s))}{\gamma[1+uv\cos(\varphi+\tilde{\theta}(\varphi+\pi,s))]}, 0\Big) : \varphi\in[0,2\pi)\Big\}.$$

Now, suppose that $\tilde{\theta}(\varphi,s_0)\neq\tilde{\theta}(\varphi+\pi,s_0)$ for some φ and s_0; then the set $\eta_{s_0}(g\mathbb{B})$, which is the support of gS_{s_0}, is different from the set (3), which is the support of gS_∞, and therefore $gS_{s_0}\neq gS_\infty$. From the equality (5) one deduces that $\eta_{s_0}(g\mathbb{B}) = \eta_{s_n}(g\mathbb{B})$ for $n=1,2,\ldots$, where $s_n := se^{2n\pi}$. As a consequence if S_s converges for $s\to\infty$ its limit cannot be gS_∞.

In order to prove the thesis, we have still to prove that there exists some φ and s_0 such that $\tilde{\theta}(\varphi,s_0)\neq\tilde{\theta}(\varphi+\pi,s_0)$. This is trivial: chose φ and θ such that $\varphi+\theta\neq(2n+1)\pi/2$, and define $s_0 := \gamma e^\theta[1-uv\cos(\varphi+\theta)]$. This implies that $\tilde{\theta}(\varphi,s_0)=\theta$; suppose that $\tilde{\theta}(\varphi,s_0)=\theta$ as well; this implies in turn that

$$\frac{\gamma e^\theta}{\omega}[1-uv\cos(\varphi+\theta)] = s_0 = \frac{\gamma e^\theta}{\omega}[1-uv\cos(\varphi+\pi+\theta)],$$

which is impossible if $uv\neq 0$.